# AN ALGEBRAIC SOLUTION OF THE EINSTEIN EQUATIONS*

BY

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The Einstein field equations of gravitation in their cosmological form (for the case where matter is not present) may be written

(1) $$R_{ik} - \lambda g_{ik} = 0$$

or, what is equivalent,

(2) $$R_{ik} - \frac{1}{4} g_{ik} R = 0.$$

We wish to present here a new particular solution which is algebraic and extremely simple both in analytic and geometric form, namely

(3) $$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

It is in fact the simplest solution beyond the hypersphere (De Sitter's solution)

(4) $$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2).$$

Our starting point is to assume the quaternary form

$$g_{ik}\, dx_i\, dx_k$$

to be the sum of two binary forms, one in the variables $x_1$, $x_2$, the other in the variables $x_3$, $x_4$.

We have then

(5) $$ds^2 = E dx_1^2 + 2 F dx_1 dx_2 + G dx_2^2 + E' dx_3^2 + 2 F' dx_3 dx_4 + G' dx_4^2$$

where $E$, $F$, $G$ involve only $x_1$, $x_2$ and $E'$, $F'$, $G'$ involve only $x_3$, $x_4$. By a transformation of variables we may, without loss of generality, assume

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$$F = 0, \qquad E = G = \mu(x_1, x_2),$$
$$F' = 0, \qquad E' = G' = \nu(x_3, x_4),$$

so that

(6) $\qquad ds^2 = \mu(x_1, x_2)\,(dx_1^2 + dx_2^2) + \nu(x_3, x_4)\,(dx_3^2 + dx_4^2).$

The problem is to find the two functions $\mu$ and $\nu$ so that equations (2) are satisfied. It is convenient to introduce

(6') $\qquad\qquad \alpha = \dfrac{1}{2}\log\mu, \qquad \beta = \dfrac{1}{2}\log\nu.$

The components of the contracted curvature tensor are easily found to be*

$$R_{11} = R_{22} = \alpha_{11} + \alpha_{22},$$
$$R_{33} = R_{44} = \beta_{33} + \beta_{44},$$

where subscripts applied to $\alpha$ and $\beta$ denote partial derivatives $\alpha_{11} = \partial^2\alpha/\partial x_1^2$, etc. The other components $R_{12}$, etc., vanish identically.

The scalar curvature is

$$R = 2\,\frac{\alpha_{11} + \alpha_{22}}{\mu} + 2\,\frac{\beta_{33} + \beta_{44}}{\nu}.$$

Substituting in (2) we find merely one condition, which may be written

$$\frac{\alpha_{11} + \alpha_{22}}{\mu} = \frac{\beta_{33} + \beta_{44}}{\nu}.$$

We observe that the left member is a function of $x_1$, $x_2$ and the right member is a function of $x_3$, $x_4$, therefore both members are equal to a constant, that is, using (6'),

$$e^{-2\alpha}\,(\alpha_{11} + \alpha_{22}) = c, \qquad e^{-2\beta}\,(\beta_{33} + \beta_{44}) = c.$$

This means that the curvature of each of the two binary forms (surfaces) is constant. It follows that each may be assumed as the first fundamental quadratic form of a sphere (radius $= c^{-1/2}$). It is, therefore, unnecessary

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* We may conveniently apply the formulas given on p. 229 of the author's paper *The solar gravitational field completely determined by its light rays*, Mathematische Annalen, vol. 85 (1922), pp. 227-236.

actually to write out the general solutions of the above partial differential equations.   It is sufficient to assume the particular solution

$$2\,\alpha \;=\; \log\frac{1}{c\,x_1^2}, \qquad 2\,\beta \;=\; \log\frac{1}{c\,x_3^2}\,.$$

Our quaternary form (6) is thus

$$(7) \qquad\qquad ds^2 \;=\; \frac{dx_1^2 + dx_2^2}{c\,x_1^2} + \frac{dx_3^2 + dx_4^2}{c\,x_3^2}\,.$$

If the constant $c$ is zero we may take $\alpha = 0$, $\beta = 0$, that is, $\mu = 1$, $\nu = 1$, so that

$$ds^2 \;=\; dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

which is euclidean 4-space.   Excluding this trivial case every solution of our problem is equivalent to (3) under a homothetic transformation.

*If the sum of two independent binary forms is to satisfy* (2) *then the forms represent equal spheres and the result is reducible to* (3).

Using another familiar element of a sphere we may write the form (which is equivalent to (7) with $c = 4\,a$)

$$(8) \qquad\qquad ds^2 \;=\; \frac{dx_1^2 + dx_2^2}{[1 + a\,(x_1^2 + x_2^2)]^2} + \frac{dx_3^2 + dx_4^2}{[1 + a\,(x_3^2 + x_4^2)]^2}\,.$$

Similarly the four-dimensional hypersphere (4) may also be written in the form

$$(8') \qquad\qquad ds^2 \;=\; \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2}{[1 + a\,(x_1^2 + x_2^2 + x_3^2 + x_4^2)]^2}\,.$$

*Thus* (8) *is an exact solution of the field equations* (2) *as well as* (8').

We may interpret our result geometrically in space of six dimensions as follows.

Take a flat space with six cartesian coördinates $X_1$, $X_2$, $X_3$, $X_4$, $X_5$, $X_6$. In the 3-flat $X_1$, $X_2$, $X_3$, take a unit sphere

$$(9') \qquad\qquad X_1^2 + X_2^2 + X_3^2 \;=\; 1,$$

and in the 3-flat $X_4$, $X_5$, $X_6$, another unit sphere

$$(9'') \qquad\qquad X_4^2 + X_5^2 + X_6^2 \;=\; 1.$$

On these spheres as bases construct hypercylinders of five dimensions. The equations of these cylinders are (9′) and (9″). The intersection of these cylinders is the four-dimensional manifold defined by the simultaneous equations

(10) $$X_1^2 + X_2^2 + X_3^2 = 1, \qquad X_4^2 + X_5^2 + X_6^2 = 1.$$

*This is an algebraic four-dimensional manifold of fourth degree which obeys the field equations* (2).

Any four-dimensional manifold in a 6-flat which is the intersection of two cylindrical 5-spreads,

(11) $$F(X_1, X_2, X_3) = 0, \qquad G(X_4, X_5, X_6) = 0,$$

will have for its $ds^2$ a quaternary form which can be written as the sum of two independent binary forms. Thus we have proved

*The only manifolds of type* (11) *in flat space of* 6 *dimensions which obey the field equations are reducible to the quartic variety* (10).

## Separable forms

A quadratic differential form in $n$ variables may be called *separable* if it can be reduced by any transformation of the variables to the sum of two forms one involving $h$ variables and the other involving $k$ variables, where $h + k = n$.

This we shall call *separable of type* $(h, k)$. The various types for a given $n$ are not necessarily mutually exclusive. There are possibilities of certain special forms belonging to more than one type. We may of course also have separable forms of type $(h_1, h_2, \cdots, h_r)$, where $h_1 + h_2 + \cdots + h_r = n$. If the form is euclidean it is of type $(1, 1, \cdots, 1)$, and vice versa this type is obviously euclidean. This is the extreme case of separability, the $n$-ary form being then transformable into the sum of $n$ independent unary forms.

If $n = 4$, the possible types are

(*a*) $(2, 2)$,
(*b*) $(3, 1)$,
(*c*) $(2, 1, 1)$,
(*d*) $(1, 1, 1, 1)$.

When can an Einstein manifold be of one of these types? For (*a*) the result has been given above. For (*b*) no solutions exist; the proof is easy but is here omitted.

For (*c*) no solutions exist except of course those which are of the trivial euclidean type (*d*). Hence we have the theorem

*If an Einstein manifold is to be separable it must be either euclidean or equivalent to* (3), *that is the quartic manifold* (10).

The separability so far defined has been *complete*. There is another more general theory of *incomplete or partial separability*. Thus a form in four variables $x_1$, $x_2$, $x_3$, $x_4$ may in certain cases, even though not separable in the first sense, be reducible for example to the sum of three binary forms

$$(12) \qquad\qquad ds^2 = Q' + Q'' + Q''',$$

where $Q'$ involves $x_1$, $x_2$, $Q''$ involves $x_1$, $x_3$, and $Q'''$ involves $x_1$, $x_4$. This refers of course to the coefficients as well as the differentials, that is

$$Q' = g_{11}\, dx_1^2 + 2\, g_{12}\, dx_1\, dx_2 + g_{22}\, dx_2^2,$$

where the $g$'s are functions of only $x_1$, $x_2$. Einstein solutions of type (12) actually exist. In particular I have found all solutions of the form*

$$(13) \qquad ds^2 = \alpha(x_1)\, dx_1^2 + \beta(x_1)\, dx_2^2 + \gamma(x_1)\, dx_3^2 + \delta(x_1)\, dx_4^2.$$

These are included in the type (12). They may be immersed in a flat space of seven dimensions and defined in finite form by means of three surfaces of rotation having a common axis.

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* See Science, vol. 54 (1921), p. 304, and American Journal of Mathematics, vol. 43 (1921), p. 220; also a forthcoming paper in these Transactions.

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